Introduction to reinforcement learning

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Outline

Introduction to the reinforcement learning framework

Artificial autonomous intelligent agent: formal definition

Definition (Agent)
An agent is anything that is capable of acting upon information it perceives.

Definition (Intelligent agent)
An intelligent agent is an agent capable of making decisions about how it acts based on experience, that is of learning decision from experience.

Definition (Autonomous intelligent agent)
An autonomous intelligent agent is an intelligent agent that is free to choose between different actions.

Definition (Artificial autonomous intelligent agent)
An artificial autonomous intelligent agent is anything we create that is capable of actions based on information it perceives, its own experience, and its own decisions about which actions to perform.

Since "artificial autonomous intelligent agent" is quite mouthful, we follow the convention of using "intelligent agent" or "autonomous agent" for short.

Application of intelligent agents

Intelligent agents are applied in a variety of areas: project management, electronic commerce, robotics, information retrieval, military, networking, planning and scheduling, etc.

Examples:

- A predictive maintenance agent for industrial equipment that analyzes sensor data to predict failures before they happen, scheduling maintenance only when needed and reducing downtime and costs Leroy et al. [2023].
- An autonomous delivery drone system that optimizes delivery routes and times based on traffic, weather conditions, and customer availability, learning from each delivery to improve efficiency and customer satisfaction.
- An alignment agent fine-tunes LLMs like ChatGPT, to better match user intentions. It learns from feedback to improve question interpretation and ensure accurate, relevant responses. See lecture 11 on RL and LLMs.
- A robotic harvesting assistant that navigates through orchards, using visual recognition to identify ripe fruits and vegetables. It gently picks produce with precision, minimizing damage and waste. By learning from each harvest what conditions lead to the best yield and quality it helps farmers optimize picking schedules. See lecture 10 on robotic RL.

Machine learning and reinforcement learning: definitions

Definition (Machine learning)
Machine learning is a broad subfield of artificial intelligence is concerned with the development of algorithms and techniques that allow computers to “learn”.

Definition (Reinforcement Learning)
Reinforcement Learning (RL in short) refers to a class of problems in machine learning which postulate an autonomous agent exploring an environment in which the agent perceives information about its current state and takes actions. The environment, in return, provides a reward signal (which can be positive or negative). The agent has as objective to maximize the (expected) cumulative reward signal over the course of the interaction.

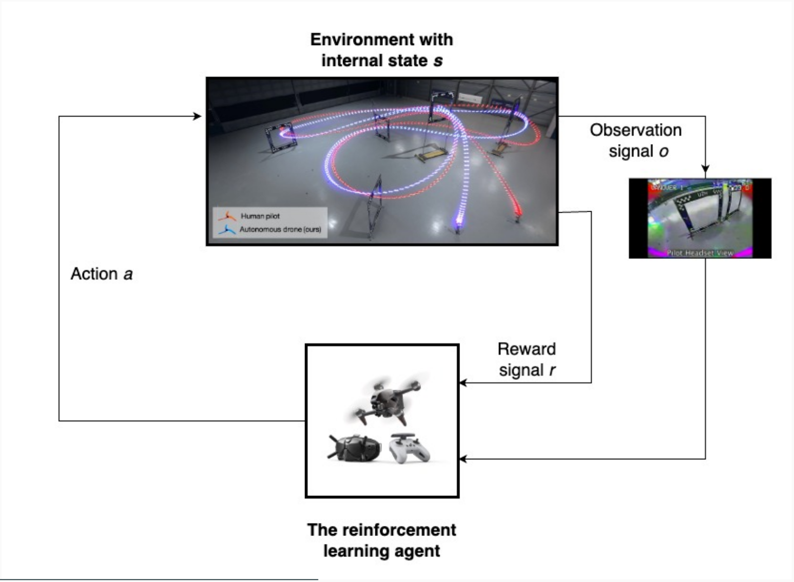
Definition (The policy)

The policy of an agent determines the way the agent selects its action based on the information it has. A policy can be either deterministic or stochastic and either stationary or history-dependant.

Research in reinforcement learning aims at designing policies which lead to large (expected) cumulative reward.

Where does the intelligence come from? The policies process in an “intelligent way” the information to select “good actions”.

An RL agent interracting with its environment



[1]

[1] Source: Kaufmann et al. [2023]

Demo

https://www.youtube.com/watch?v=EtRXay2kqtc

Some generic difficulties with designing intelligent agents

- Inference problem. The environment dynamics and the mechanism behind the reward signal are (partially) unknown. The policies need to be able to infer from the information the agent has gathered from interaction with the system, “good control actions”.

- Computational complexity. The policy must be able to process the history of the observation within limited amount of computing time and memory.

- Tradeoff between exploration and exploitation.[2] To obtain a lot of reward, a reinforcement learning agent must prefer actions that it has tried in the past and found to be effective in producing reward. But to discover such actions, it has to try actions that it has not selected before.

[2]May be seen as a subproblem of the general inference problem. This problem is often referred to in the “classical control theory” as the dual control problem.

The agent has to exploit what it already knows in order to obtain reward, but it also has to explore in order to make better action selections in the future. The dilemma is that neither exploration nor exploitation can be pursued exclusively without failing at the task. The agent must try a variety of actions and progressively favor those that appear to be best. On a stochastic task, each action must be tried many times to gain a reliable estimate its expected reward.

- Exploring safely the environment. During an exploration phase (more generally, any phase of the agent's interaction with its environment), the agent must avoid reaching unacceptable states (e.g., states that may for example endanger its own integrity). By associating rewards of $-\infty$ to those states, exploring safely can be assimilated to a problem of exploration-exploitation.

- Adversarial environment. The environment may be adversarial. In such a context, one or several other players seek to adopt strategies that oppose the interests of the RL agent.

Characterization of RL problems

Different characterizations of RL problems

- Stochastic (e.g., $s_{t+1} = f(s_t, a_t, w_t)$ where the random disturbance w_t is drawn according to the conditional probability distribution $P_w(\cdot|s_t, a_t)$) versus deterministic (e.g., $s_{t+1} = f(s_t, a_t)$)
- Partial observability versus full observability. The environment is said to be partially (fully) observable if the signal o_t describes partially (fully) the environment's state s_t at time t.
- Time-invariant (e.g., $s_{t+1} = f(s_t, a_t, w_t)$ with $w_t = P_w(\cdot|s_t, a_t)$) versus time-variant (e.g., $s_{t+1} = f(s_t, a_t, w_t, t)$) dynamics.
- Continuous (e.g., $\dot{s} = f(s, a, w)$) versus discrete dynamics (e.g., $s_{t+1} = f(s_t, a_t, w_t)$).
- Finite time versus infinite time of interaction.

- Multi-agent framework versus single-agent framework. In a multi-agent framework the environment may be itself composed of (intelligent) agents. A multi-agent framework can often be assimilated to a single-agent framework by considering that the internal states of the other agents are unobservable variables. Game theory and, more particularly, the theory of learning in games study situations where various intelligent agents interact with each other.

- Single state versus multi-state environment. In single state environment, computation of an optimal policy for the agent is often reduced to the computation of the maximum of a stochastic function (e.g., find $a^* \in \arg\max_{a \in \mathcal{A}} \underset{w \sim P_w(\cdot|a)}{E} [r(a, w)]$).

- Multi-objective reinforcement learning agent (reinforcement learning signal can be multi-dimensional) versus single-objective RL agent.

- Risk-adverse reinforcement learning agent. The goal of the agent is not anymore to maximize the expected cumulative reward but maximize the lowest cumulative reward it could possibly obtain.

Characterization of the RL problem adopted in this class

- Dynamics of the environment:

$$s_{t+1} = f(s_t, a_t, w_t) \quad t = 0, 1, 2 \ldots$$

where for all t, the state s_t is an element of the state space S, the action a_t is an element of the action space A and the random disturbance w_t is an element of the disturbance space W. Disturbance w_t generated by the time-invariant conditional probability distribution $P_w(\cdot|s, a)$.

- Reward signal:

The function $r(s, a, w)$ is the so-called reward function supposed to be bounded by below by 0 and by above by a constant $B_r \geq 0$.
To the transition from t to $t + 1$ is associated a reward signal $\gamma^t r_t = \gamma^t r(s_t, a_t, w_t)$ where $r(s, a, w)$ is a reward function supposed to be bounded by a constant B_r and $\gamma \in [0, 1[$ a decay factor.

Definition (Cumulative reward signal)

Let $h_t \in \mathcal{H}$ be the trajectory from instant time 0 to t in the combined state, action, reward spaces: $h_t = (s_0, a_0, r_0, s_1, a_1, r_1, \ldots, a_{t-1}, r_{t-1}, s_t)$. Let $\pi \in \Pi$ be a stochastic policy such that $\pi : \mathcal{H} \times \mathcal{A} \rightarrow [0, 1]$ and let us denote by $J^\pi(s)$ the expected return of a policy π (or expected cumulative reward signal) when the system starts from $s_0 = s$

$$J^\pi(s) = \lim_{T\rightarrow\infty} E[\sum_{t=0}^{T} \gamma^t r(s_t, a_t \sim \pi(h_t, .), w_t)|s_0 = s]$$

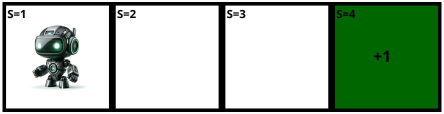
- Information available:

The agent does not know f, r and P_w. The only information it has on these three elements is the information contained in h_t.

Exercise 1: Computation of the cumulative reward signal

Compute the cumulative reward signal $J^{\pi}(1)$ with policy $\pi(s) = 1, \quad \forall s \in \mathcal{S}$.
The state space and action space are respectively defined by:

$$\mathcal{S} = \{s \in \{1,2,3,4\}\}, \quad \mathcal{A} = \{a \in \{-1,1\}\}$$



The reward function and the dynamics are the following:

$$f(s,a) = (\min(\max(s+a,1),4)$$

$$r(s,a) = \begin{cases} 1 & \text{if } f(s,a) = 4 \\ 0 & \text{otherwise} \end{cases}$$

Exercise 1: Solution

Solution:

$$\begin{aligned}J^\pi(1) =& \lim_{T\to\infty}[r(1,1) + \gamma r(2,1) + \gamma^2 r(3,1) + \gamma^3 r(4,1) + ... + \gamma^T r(4,1)]\\ =& \lim_{T\to\infty}[\gamma^2 + ... + \gamma^T]\\ =& \gamma^2 \lim_{T\to\infty}[1 + ... + \gamma^{T-2}]\\ =& \frac{\gamma^2}{1-\gamma}\end{aligned}$$

Reminder: $\lim\limits_{T\to\infty} \sum_{t=0}^{T} \gamma^t = \frac{1}{1-\gamma}$

Goal of reinforcement learning

Goal of reinforcement learning

Theorem (Optimal policy existence)

Let $\pi^* \in \Pi$ a policy such that $\forall s \in \mathcal{S}$,

$$J^{\pi^*}(s) = \max_{\pi \in \Pi} J^{\pi}(s) \tag{1}$$

Under some mild assumptions[3] *on f, r and P_w, such a policy π^* indeed exists.*

- **In reinforcement learning, we want to build policies $\hat{\pi}^*$ such that $J^{\hat{\pi}^*}$ is as close as possible (according to specific metrics) to J^{π^*}.**

- If f, r and P_w were known, we could, by putting aside the difficulty of finding in Π the policy π^*, design the optimal agent by solving the optimal control problem (1). However, J^{π} depends on f, r and P_w which are supposed to be unknown $\Rightarrow$ How can we solve this combined inference - optimization problem?

[3] We will suppose that these mild assumptions are always satisifed afterwards.

Dynamic Programming (DP) theory reminder: optimality of stationary policies

Definition (Deterministic stationary policy)

A deterministic stationary control policy $\mu : \mathcal{S} \to \mathcal{A}$ selects at time t the action $a_t = \mu(s_t)$. We denote Π_μ the set of stationary policies.

Definition (Expected return)

The expected return of a stationary policy, when the system starts from $s_0 = s$, is:

$$J^\mu(s) = \lim_{T \to \infty} \underset{w_0, w_1, \ldots, w_T}{E} [\sum_{t=0}^{T} \gamma^t r(s_t, \mu(s_t), w_t) | s_0 = s]. \tag{2}$$

Let μ^* be a policy such that $J^{\mu^*}(s) = \max_{\mu \in \Pi_\mu} J^\mu(s)$ everywhere on $\mathcal{S}$. We name such a policy an optimal deterministic stationary policy.

We denote π^* the optimal history-dependent policy, from classical dynamic programming theory, we know $J^{\mu^*}(s) = J^{\pi^*}(s)$ everywhere.
$\Rightarrow$ **considering only stationary policies is not suboptimal!**

Truncated return J_N^μ

Definition (Truncated return J_N^μ)

We define the functions $J_N^\mu : \mathcal{S} \to \mathbb{R}$ by the recurrence equation

$$J_N^\mu(s) = \underset{w \sim P_w(\cdot|s,a)}{E}[r(s, \mu(s), w) + \gamma J_{N-1}^\mu(f(s, \mu(s), w))], \quad \forall N \geq 1 \tag{3}$$

with $J_0^\mu(s) \equiv 0$.

We have as result of the dynamic programming (DP) theory

$$\lim_{N\to\infty} \|J^\mu - J_N^\mu\|_\infty \to 0. \tag{4}$$

Similarly, we can also write

$$J^\mu(s) = J_N^\mu(s) + \gamma^N J^\mu(s_N), \quad \forall s \in \mathcal{S} \tag{5}$$

where s_N is the state after N steps.
Moreover, we can show that there exists a bound on the value of $J^\mu(s)$

$$\|J^\mu\|_\infty = \lim_{N\to\infty} \sum_{t=0}^{N} \gamma^t B_r = \lim_{N\to\infty} \frac{1-\gamma^{N+1}}{1-\gamma} B_r = \frac{B_r}{1-\gamma} \tag{6}$$

where $B_r = \|r\|_\infty$ and iff $\gamma \in [0;1[$ and $r : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^+$.

Relation between J^μ and J_N^μ

Using (5) and (6), we can show that there exists a bound on $\|J^\mu - J_N^\mu\|_\infty$

$$\|J^\mu - J_N^\mu\|_\infty \leq \frac{\gamma^N}{1-\gamma} B_r. \tag{7}$$

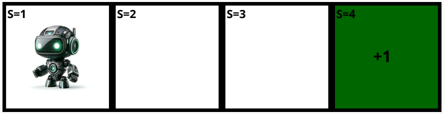
Indeed,

$$\|J^\mu - J_N^\mu\|_\infty \leq \gamma^N \|J^\mu\|_\infty = \frac{\gamma^N}{1-\gamma} B_r.$$

Exercise 2: Computation of the J_N^μ function

Compute $J_4^\mu(1)$, $J_4^\mu(2)$, $J_4^\mu(3)$ and $J_4^\mu(4)$ with policy $\pi(s) = 1, \quad \forall s \in \mathcal{S}$.
The state space and action space are respectively defined by:

$$\mathcal{S} = \{s \in \{1,2,3,4\}\}, \quad \mathcal{A} = \{a \in \{-1,1\}\}$$



The reward function and the dynamics are the following:

$$f(s,a) = (\min(\max(s+a,1),4)$$

$$r(s,a) = \begin{cases} 1 & \text{if } f(s,a) = 4 \\ 0 & \text{otherwise} \end{cases}$$

Exercise 2: Solution

Solution:

$$J_N^\mu(s) = r(s, \mu(s)) + \gamma J_{N-1}^\mu(f(s, \mu(s))), \quad J_0^\mu(s) \equiv 0$$

$$\begin{aligned}
&J_1^\mu(1) = 0, \quad J_1^\mu(2) = 0, \quad J_1^\mu(3) = 1, \quad J_1^\mu(4) = 1 \\
&J_2^\mu(1) = 0, \quad J_2^\mu(2) = \gamma, \quad J_2^\mu(3) = 1 + \gamma, \quad J_2^\mu(4) = 1 + \gamma \\
&J_3^\mu(1) = \gamma^2, \quad J_3^\mu(2) = \gamma + \gamma^2, \quad J_3^\mu(3) = J_3^\mu(4) = 1 + \gamma + \gamma^2 \\
&J_4^\mu(1) = \gamma^2 + \gamma^3, \quad J_4^\mu(2) = \gamma + \gamma^2 + \gamma^3, \quad J_4^\mu(3) = J_4^\mu(4) = 1 + \gamma + \gamma^2 + \gamma^3
\end{aligned}$$

DP theory reminder: Q_N-functions and Bellman equation

Definition (Q_N-functions)

We define the functions $Q_N : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ by the recurrence equation

$$Q_N(s,a) = \underset{w \sim P_w(\cdot|s,a)}{E}[r(s,a,w) + \gamma \max_{a' \in \mathcal{A}} Q_{N-1}(f(s,a,w),a')], \quad \forall N \geq 1 \tag{8}$$

with $Q_0(s,a) \equiv 0$. These Q_N-functions are also known as state-action value functions.

Definition (Q-function)

We define the Q-function as being the unique solution of the Bellman equation:

$$Q(s,a) = \underset{w \sim P_w(\cdot|s,a)}{E}[r(s,a,w) + \gamma \max_{a' \in \mathcal{A}} Q(f(s,a,w),a')]. \tag{9}$$

Theorem (Convergence of Q_N)

The sequence of functions Q_N converges to the Q-function in the infinite norm, i.e. $\lim_{N \to \infty} \|Q_N - Q\|_\infty \to 0$.

Exercise 3: Computation of the Q_N function

Compute $Q_4(s, a), \quad \forall (s, a) \in \mathcal{S} \times \mathcal{A}$ with policy $\pi(s) = 1, \quad \forall s \in \mathcal{S}$.
The state space and action space are respectively defined by:

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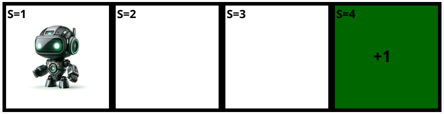


The reward function and the dynamics are the following:

$$f(s, a) = (\min(\max(s + a, 1), 4)$$

$$r(s, a) = \begin{cases} 1 & \text{if } f(s, a) = 4 \\ 0 & \text{otherwise} \end{cases}$$

Solution

Solution:

$$Q_N(s,a) = r(s,a) + \gamma \max_{a' \in \mathcal{A}} Q_{N-1}(f(s,a),a'), \quad Q_0(s,a) \equiv 0$$

N	1	2	3	4	5
$Q_N(1,-1)$	0	0	0	γ^3	$\gamma^3+\gamma^4$
$Q_N(1,1)$	0	0	γ^2	$\gamma^2+\gamma^3$	$\gamma^2+\gamma^3+\gamma^4$
$Q_N(2,-1)$	0	0	0	γ^3	$\gamma^3+\gamma^4$
$Q_N(2,1)$	0	γ	$\gamma+\gamma^2$	$\gamma+\gamma^2+\gamma^3$	$\gamma+\gamma^2+\gamma^3+\gamma^4$
$Q_N(3,-1)$	0	0	γ^2	$\gamma^2+\gamma^3$	$\gamma^2+\gamma^3+\gamma^4$
$Q_N(3,1)$	1	$1+\gamma$	$1+\gamma+\gamma^2$	$1+\gamma+\gamma^2+\gamma^3$	$1+\gamma+\gamma^2+\gamma^3+\gamma^4$
$Q_N(4,-1)$	0	γ	$\gamma+\gamma^2$	$\gamma+\gamma^2+\gamma^3$	$\gamma+\gamma^2+\gamma^3+\gamma^4$
$Q_N(4,1)$	1	$1+\gamma$	$1+\gamma+\gamma^2$	$1+\gamma+\gamma^2+\gamma^3$	$1+\gamma+\gamma^2+\gamma^3+\gamma^4$

Optimal stationary policy

Theorem (Optimal stationary policy)

A stationary policy μ^* is optimal if and only if

$$\mu^*(s) \in \arg\max_{a \in A} Q(s,a) \tag{10}$$

We also have $J^{\mu^*}(s) = \max_{a \in \mathcal{A}} Q(s,a)$, which is also called the value function.

Theorem (N-optimal stationary policy)

A stationary policy μ_N^* is N-optimal iff it selects an optimal action when there remains exactly N steps.

$$\mu_N^*(s) \in \arg\max_{a \in \mathcal{A}} Q_N(s,a) \tag{11}$$

Necessarily, μ_N^* is suboptimal with respect to μ^*, i.e.,

$$J^{\mu^*}(s) \geq J^{\mu_N^*}(s) \tag{12}$$

Bound on the suboptimality of $J^{\mu_N^*}$

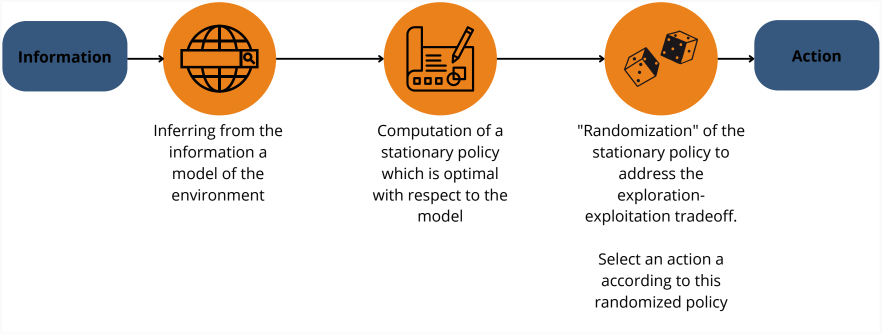
Theorem (Bound on the suboptimality of $J^{\mu_N^*}$)

There exists a bound on the suboptimality of μ_N^* in comparison to μ^*, which is given by the following inequality:

$$\left\| J^{\mu^*} - J^{\mu_N^*} \right\|_\infty \leq \frac{2\gamma^N B_r}{(1-\gamma)^2} \tag{13}$$

RL problems with small state-action spaces

A pragmatic model-based approach for designing good policies $\hat{\pi}^*$


Information
Inferring from the information a model of the environment
Computation of a stationary policy which is optimal with respect to the model
"Randomization" of the stationary policy to address the exploration-exploitation tradeoff.
Select an action a according to this randomized policy
Action

We focus first on to the design of functions $\hat{\pi}^*$ which realize sequentially the following three tasks:

1. "System identification" phase. Estimation from h_t of an approximate system dynamics $\hat{f}$, an approximate probability distribution $\hat{P}_w$ and an approximate reward function $\hat{r}$.

2. Resolution of the optimization problem.

Find in Π_μ the policy $\hat{\mu}^*$ such that $\forall s \in \mathcal{S}, J^{\hat{\mu}^*}(s) = \max_{\mu \in \Pi_\mu} \hat{J}^{\mu}(s)$
where $\hat{J}^{\hat{\mu}}$ is defined similarly as function J^{μ} but with $\hat{f}$, $\hat{P}_w$ and $\hat{r}$ replacing f, P_w and r, respectively.

3. Afterwards, the policy $\hat{\pi}$ selects with a probability $1 - \varepsilon(h_t)$ actions according to the policy $\hat{\mu}^*$ and with a probability $\varepsilon(h_t)$ at random. Step 3 has been introduced to address the dilemma between exploration and exploitation.[4]

[4] We will not address further the design of the 'right function' $\varepsilon : \mathcal{H} \to [0, 1]$. In many applications, it is chosen equal to a small constant (say, 0.05) everywhere.

Some constructive algorithms for designing $\hat{\pi}^*$ when dealing with finite state-action spaces

- Until say otherwise, we consider the particular case of finite state and action spaces (i.e., $\mathcal{S} \times \mathcal{A}$ finite).

- When $\mathcal{S}$ and $\mathcal{A}$ are finite, there exists a vast panel of 'well-working' implementable RL algorithms.

- We focus first on approaches which solve separately Step 1. and Step 2. and then on approaches which solve both steps together.

- The proposed algorithms infer $\hat{\mu}^*$ from h_t. They can be adapted in a straigthforward way to episode-based reinforcement learning where a model of μ^* must be inferred from several trajectories $h_{t_1}, h_{t_2}, \ldots, h_{t_m}$ with $t_i \in \mathbb{N}_0$.

Reminder on Markov Decision Processes

Definition (Markov Decision Process)

A Markov Decision Process (MDP) is defined through the following objects: a state space $\mathcal{S}$, an action space $\mathcal{A}$, transition probabilities $p(s'|s,a)$ $\forall s, s' \in \mathcal{S}$, $a \in \mathcal{A}$ and a reward function $r(s,a)$.

- $p(s'|s,a)$ gives the probability of reaching state s' after taking action a while being in state s.

- We consider MDPs for which we want to find decision policies that maximize the reward signal $\gamma^t r(s_t, a_t)$ over an infinite time horizon.

- MDPs can be seen as a particular type of the discrete-time optimal control problem introduced earlier.

MDP Structure Definition from the System Dynamics and Reward Function

- We define[5]

$$r(s,a) = \underset{w \sim P_w(\cdot|s,a)}{E}[r(s,a,w)] \quad \forall s \in \mathcal{S}, a \in \mathcal{A} \tag{14}$$

$$p(s'|s,a) = \underset{w \sim P_w(\cdot|s,a)}{E}[I_{\{s'=f(s,a,w)\}}] \quad \forall s, s' \in \mathcal{S}, a \in \mathcal{A} \tag{15}$$

- Equations (14) and (15) define the structure of an equivalent MDP in the sense that the expected return of any policy applied to the original optimal control problem is equal to its expected return for the MDP.

- The recurrence equation defining the functions Q_N can be rewritten: $Q_N(s,a) = r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) \max_{a' \in \mathcal{A}} Q_{N-1}(s',a'), \quad \forall N \geq 1$ with $Q_0(s,a) \equiv 0$.

[5] $I_{\{logical\_expression\}} = 1$ if $logical\_expression$ is $true$ and 0 if $logical\_expression$ is $false$.

Reminder: Random Variable and Strong Law of Large Numbers

- A random variable is not a variable but rather a function that maps outcomes (of an experiment) to numbers. Mathematically, a random variable is defined as a measurable function from a probability space to some measurable space. We consider here random variables θ defined on the probability space (Ω, P).[6]

- $\underset{P}{E}[\theta]$ is the mean value of the random variable θ.

- Let θ_1, θ_2, $\ldots$, θ_2 be n values of the random variable θ which are drawn independently. Suppose also that $\underset{P}{E}[|\theta|] = \int_\Omega |\theta| dP$ is smaller than ∞. In such a case, the strong law of large number states that:

$$\lim_{n\to\infty} \frac{\theta_1 + \theta_2 + \ldots + \theta_n}{n} \xrightarrow{P} \underset{P}{E}[\theta] \tag{16}$$

[6]For the sake of simplicity, we have considered here that (Ω, P) indeed defines a probability space which is not rigorous.

Step 1. Identification by learning the structure of the equivalent MPD

- The objective is to infer some 'good approximations' of $p(s'|s,a)$ and $r(s,a)$ from:

$$h_t = (s_0, a_0, r_0, s_1, a_1, r_1, \ldots, a_{t-1}, r_{t-1}, s_t).$$

Estimation of $r(s,a)$:

Let $X(s,a) = \{k \in \{0,1,\ldots,t-1\} | (s_k, a_k) = (s,a)\}$. Let k_1, k_2, ..., $k_{\#X(s,a)}$ denote the elements of the set.[7] The values r_{k_1}, r_{k_2}, ..., $r_{k_{\#X(s,a)}}$ are $\#X(s,a)$ values of the random variable $r(s,a,w)$ which are drawn independently. It follows therefore naturally that to estimate its mean value $r(s,a)$, we can use the following unbiased estimator:

$$\hat{r}(s,a) = \frac{\sum_{k \in X(s,a)} r_k}{\#X(s,a)} \tag{17}$$

[7] If X is a set of elements, $\#X$ denote the cardinality of X.

Estimation of $p(s'|s,a)$:

The values $I_{\{s'=s_{k_1+1}\}}$, $I_{\{s'=s_{k_2+1}\}}$, . . ., $I_{\{s'=s_{k_{\#X(s,a)}+1}\}}$ are $\#X(s,a)$ values of the random variable $I_{\{s'=f(s,a,w)\}}$ which are drawn independently. To estimate its mean value $p(s'|s,a)$, we can use the unbiased estimator:

$$\hat{p}(s'|s,a) = \frac{\sum_{k \in X(s,a)} I_{\{s_{k+1}=s'\}}}{\#X(s,a)} \tag{18}$$

Step 2. Computation of $\hat{\mu}^*$ identification by learning the structure of the equivalent MPD

• We compute the $\hat{Q}_N$-functions from the knowledge of $\hat{r}$ and $\hat{p}$ by exploiting the recurrence equation:

$\hat{Q}_N(s,a) = \hat{r}(s,a) + \gamma \sum_{s' \in S} \hat{p}(s'|s,a) \max_{a' \in A} Q_{N-1}(s',a'), \quad \forall N \geq 1$ with $\hat{Q}_0(s,a) \equiv 0$ and then take

$$\hat{\mu}_N^* = \arg\max_{a \in A} \hat{Q}_N(s,a) \quad \forall s \in S \tag{19}$$

as approximation of the optimal policy, with N 'large enough' (e.g., right hand side of inequality (13) drops below ε).

• One can show that if the estimated MDP structure lies in an 'ε-neighborhood' of the true structure, then, $J^{\hat{\mu}^*}$ is in a '$O(\varepsilon)$-neighborhood' of J^{μ^*} where $\hat{\mu}^*(s) = \lim_{N \to \infty} \arg\max_{a \in A} \hat{Q}_N(s,a)$.

The Case of Limited Computational Resources

- Number of operations to estimate the MDP structure grows linearly with t. Memory requirements needed to store h_t also grow linearly with t $\Rightarrow$ an agent having limited computational resources will face problems after certain time of interaction.

- We describe an algorithm which requires at time t a number of operations that does not depend on t to update the MDP structure and for which the memory requirements do not grow with t:

At time 0, set $N(s,a) = 0$, $N(s,a,s') = 0$, $R(s,a) = 0$, $p(s'|s,a) = 0$, $\forall s, s' \in \mathcal{S}$ and $a \in \mathcal{A}$.

At time $t \neq 0$, do

1. $N(s_{t-1}, a_{t-1}) \leftarrow N(s_{t-1}, a_{t-1}) + 1$
2. $N(s_{t-1}, a_{t-1}, s_t) \leftarrow N(s_{t-1}, a_{t-1}, s_t) + 1$
3. $R(s_{t-1}, a_{t-1}) \leftarrow R(s_{t-1}, a_{t-1}) + r_t$
4. $r(s_{t-1}, a_{t-1}) \leftarrow \frac{R(s_{t-1}, a_{t-1})}{N(s_{t-1}, a_{t-1})}$
5. $p(s|s_{t-1}, a_{t-1}) \leftarrow \frac{N(s_{t-1}, a_{t-1}, s)}{N(s_t, a_t)} \quad \forall s \in \mathcal{S}$

The Q-learning Algorithm

Idea: merge steps 1 and 2 to learn directly the Q-function.

The Q-learning algorithm is an algorithm that infers directly from

$$h_t = (s_0, a_0, r_0, s_1, a_1, r_1, \ldots, a_{t-1}, r_{t-1}, s_t)$$

an approximate value of the Q-function, without identifying the structure of a Markov Decision Process.

The algorithm can be described by the following steps:

1. Initialisation of $\hat{Q}(s, a)$ to 0 everywhere. Set $k = 0$.
2. $\hat{Q}(s_k, a_k) \leftarrow (1 - \alpha_k)\hat{Q}(s_k, a_k) + \alpha_k(r_k + \gamma \max_{a \in A} \hat{Q}(s_{k+1}, a))$
3. $k \leftarrow k + 1$. If $k = t$, return $\hat{Q}$ and stop. Otherwise, go back to 2.

Q-learning: some remarks

- Iteration 2. can be rewritten as $\hat{Q}(s_k, a_k) \leftarrow \hat{Q}(s_k, a_k) + \alpha_k \delta(s_k, a_k)$ where the term:

$$\delta(s_k, a_k) \quad = \quad r_k + \gamma \max_{a \in \mathcal{A}} \hat{Q}(s_{k+1}, a) - \hat{Q}(s_k, a_k), \qquad (20)$$

called the *temporal difference*.

- Learning ratio α_k: The learning ratio α_k is often chosen constant with k and equal to a small value (e.g., $\alpha_k = 0.05, \forall k$).

- Consistency of the Q-learning algorithm: Under some particular conditions on the way α_k decreases to zero ($\lim_{t \to \infty} \sum_{k=0}^{t-1} \alpha_k \to \infty$ and $\lim_{t \to \infty} \sum_{k=0}^{t-1} \alpha_k^2 < \infty$) and the history h_t (when $t \to \infty$, every state-action pair needs to be visited an infinite number of times), $\hat{Q} \to Q$ when $t \to \infty$. (e.g. $\alpha_k = \frac{1}{k}$)

- Experience replay: At each iteration, the Q-learning algorihtm uses a sample $l_k = (s_k, a_k, r_k, s_{k+1})$ to update the function $\hat{Q}$. If rather that to use the finite sequence of sample l_0, l_2, ..., l_{t-1}, we use the infinite size sequence l_{i_1}, l_{i_2}, ... to update in a similar way $\hat{Q}$, where the i_j are i.i.d. with uniform distribution on $\{0, 2, \ldots, t-1\}$, then $\hat{Q}$ converges to the approximate Q-function computed from the estimated equivalent MDP structure.

RL problems with large state-action spaces

Inferring $\hat{\mu}^*$ from h_t when dealing with very large or infinite state-action spaces

- Up to now, we have considered problems having discrete (and not too large) state and action spaces $\Rightarrow$ $\hat{\mu}^*$ and the $\hat{Q}_N$-functions could be represented in a tabular form.
- We consider now the case of very large or infinite state-action spaces: functions approximators need to be used to represent $\hat{\mu}^*$ and the $\hat{Q}_N$-functions.
- These function approximators need to be used in a way that there are able to 'well generalize' over the whole state-action space the information contained in h_t.
- There is a vast literature on function approximators in reinforcement learning. We focus first on one algorithm named 'fitted Q iteration' which computes the functions $\hat{Q}_N$ from h_t by solving a sequence of batch mode supervised learning problems.

Reminder: Batch mode supervised learning

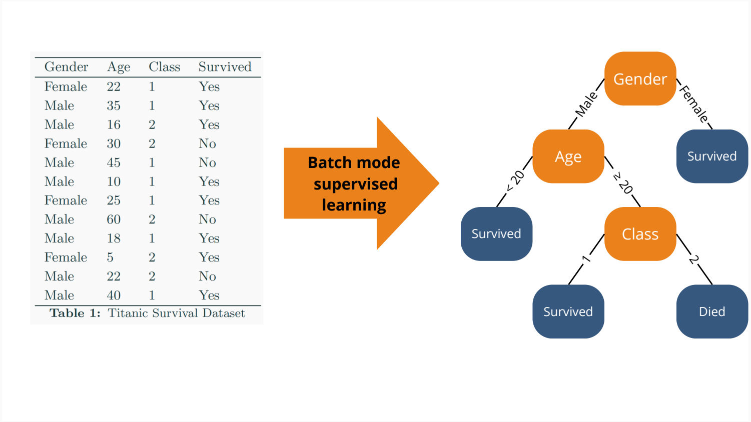
- A batch mode Supervised Learning (SL) algorithm infers from a set of input-output (input = information state); (output = class label, real number, graph, etc) a model which explains “at best” these input-output pairs.

- A loose formalisation of the SL problem: Let I be the input space, O the output space, Ξ the disturbance space. Let $g : I \times \Xi \rightarrow O$. Let $P_\xi(\cdot|i)$ a conditional probability distribution over the disturbance space.

We assume that we have a training set $\mathcal{T}S = \{(i^l, o^l)\}_{l=1}^{\#\mathcal{T}S}$ such that o^l has been generated from i^l by the following mechanism: draw $\xi \in \Xi$ according to $P_\xi(\cdot|i^l)$ and then set $o^l = g(i^l, \xi)$.

From the sole knowledge of $\mathcal{T}S$, supervised learning aims at finding a function $\hat{\bar{g}} : I \rightarrow O$ which is a ‘good approximation’ of the function $\bar{g}(i) = \underset{\xi \sim P_\xi(\cdot)}{E}[g(i, \xi)]$

• Typical supervised learning methods are: kernel-based methods, (deep) neural networks, tree-based methods.



Gender	Age	Class	Survived
Female	22	1	Yes
Male	35	1	Yes
Male	16	2	Yes
Female	30	2	No
Male	45	1	No
Male	10	1	Yes
Female	25	1	Yes
Male	60	2	No
Male	18	1	Yes
Female	5	2	Yes
Male	22	2	No
Male	40	1	Yes

Table 1: Titanic Survival Dataset

• Supervised learning highly successful: state-of-the art SL algorithms have been successfully applied to problems where the input state was composed thousands of components.

The fitted Q iteration algorithm

• Fitted Q iteration computes from h_t the functions $\hat{Q}_1$, $\hat{Q}_2$, ..., $\hat{Q}_N$, approximations of Q_1, Q_2, ..., Q_N. At step $N > 1$, the algorithm uses the function $\hat{Q}_{N-1}$ together with h_t to compute a new training set from which a SL algorithm outputs $\hat{Q}_N$. More precisely, this iterative algorithm works as follows:

First iteration: the algorithm determines a model $\hat{Q}_1$ of $Q_1(s,a) = \underset{w \sim P_w(\cdot|s,a)}{E}[r(s,a,w)]$ by running a SL algorithms on the training set:

$$\mathcal{T}S = \{((s_k, a_k), r_k)\}_{k=0}^{t-1} \tag{21}$$

Motivation: One can assimilate $S \times A$ to I, $\mathbb{R}$ to O, W to Ξ, $P_w(\cdot|s,a)$ to $P_\xi(\cdot|s,a)$, $r(s,a,w)$ to $g(i,\xi)$ and $Q_1(s,a)$ to $\overline{g}$. From there, we can observe that a SL algorithm applied to the training set described by equation (21) will produce a model of Q_1.

Iteration $N > 1$: the algorithm outputs a model $\hat{Q}_N$ of $Q_N(s,a) = \underset{w \sim P_w(\cdot|s,a)}{E}[r(s,a,w) + \gamma \max_{a' \in \mathcal{A}} Q_{N-1}(f(s,a,w),a')]$ by running a SL algorithms on the training set:

$$\mathcal{T}S = \{((s_k,a_k), r_k + \gamma \max_{a' \in \mathcal{A}} \hat{Q}_{N-1}(s_{k+1},a')\}_{k=0}^{t-1}$$

Motivation: One can reasonably suppose that $\hat{Q}_{N-1}$ is a a sufficiently good approximation of Q_{N-1} to be consider to be equal to this latter function. Assimilate $\mathcal{S} \times \mathcal{A}$ to I, $\mathbb{R}$ to O, W to Ξ, $P_w(\cdot|s,a)$ to $P_\xi(\cdot|s,a)$, $r(s,a,w)$ to $g(i,\xi)$ and $Q_N(s,a)$ to $\overline{g}$. From there, we observe that a SL algorithm applied to the training set described by equation (22) will produce a model of Q_N.

- The algorithm stops when N is ‘large enough’ and $\hat{\mu}_N^*(s) \in \arg\max_{a \in \mathcal{A}} \hat{Q}_N(s,a)$ is taken as approximation of $\mu^*(s)$.

The fitted Q iteration algorithm: some remarks

- Performances of the algorithm depends on the supervised learning (SL) method chosen.

- Excellent and stable performances have been observed when combined with supervised learning methods based on ensemble of regression trees and of course, with deep neural nets, especially when images are used as input.

- Fitted Q iteration algorithm can be used with any set of one-step system transitions (s_t, a_t, r_t, s_{t+1}) where each one-step system transition gives information about: a state, the action taken while being in this state, the reward signal observed and the next state reached.

- Consistency, that is convergence towards an optimal solution when the number of one-step system transitions tends to infinity, can be ensured under appropriate assumptions on the SL method, the sampling process, the system dynamics and the reward function.

Computation of $\hat{\mu}^*$: from an inference problem to a problem of computational complexity

- When having at one's disposal only a few one-step system transitions, the main problem is a problem of inference.

- Computational complexity of the fitted Q iteration algorithm grows with the number M of one-step system transitions (s_k, a_k, r_k, s_{k+1}) (e.g., it grows as $M \log M$ when coupled with tree-based methods).

- Above a certain number of one-step system transitions, a problem of computational complexity appears.

- Should we rely on algorithms having less inference capabilities than the 'fitted Q iteration algorithm' but which are also less computationally demanding to mitigate this problem of computational complexity $\Rightarrow$ Open research question.

- There is a serious problem plaguing every reinforcement learning algorithm known as the curse of dimensionality[8]: whatever the mechanism behind the generation of the trajectories and without any restrictive assumptions on $f(s, a, w)$, $r(s, a, w)$, S and A, the number of computer operations required to determine (close-to-) optimal policies tends to grow exponentially with the dimensionality of $\mathcal{S} \times \mathcal{A}$.

- This exponentional growth makes these techniques rapidly computationally impractical when the size of the state-action space increases.

- Many researchers in reinforcement learning/dynamic programming/optimal control theory focus their effort on designing algorithms able to break this curse of dimensionality. Deep neural nets give strong hopes for some classes of problems.

[8]A term introduced by Richard Bellman (the founder of the DP theory) in the fifties.

Q-learning with parametric function approximators

Let us extend the Q-learning algorithm to the case where a parametric Q-function of the form $\tilde{Q}(s, a, \theta)$ is used:

1. Equation (20) provides us with a desired update for $\tilde{Q}(s_t, a_t, \theta)$, here: $\delta(s_t, a_t) = r_t + \gamma \max_{a \in \mathcal{A}} \hat{Q}(s_{t+1}, a, \theta) - \hat{Q}(s_t, a_t, \theta)$, after observing (s_t, a_t, r_t, s_{t+1}).

2. It follows the following change in parameters:

$$\theta \leftarrow \theta + \alpha \delta(s_t, a_t) \frac{\partial \tilde{Q}(s_t, a_t, \theta)}{\partial \theta}. \tag{22}$$

Convergence of Q-learning

Contraction mapping

Let $B(E)$ be the set of all bounded real-valued functions defined on an arbitrary set E. With every function $R : E \to \mathbb{R}$ that belongs to $B(E)$, we associate the scalar:

$$\|R\|_\infty = \sup_{e \in E} |R(e)|. \tag{23}$$

A mapping $G : B(E) \to B(E)$ is said to be a *contraction mapping* if there exists a scalar $\rho < 1$ such that:

$$\|GR - GR'\|_\infty \leq \rho \|R - R'\|_\infty \quad \forall R, R' \in B(E). \tag{24}$$

Fixed point

$R^* \in B(E)$ is said to be a *fixed point* of a mapping $G : B(E) \to B(E)$ if:

$$GR^* = R^*. \tag{25}$$

If $G : B(E) \to B(E)$ is a *contraction mapping* then there exists a unique fixed point of G. Furthermore if $R \in B(E)$, then

$$\lim_{k\to\infty} \|G^k R - R^*\|_\infty = 0. \tag{26}$$

From now on, we assume that:

1. E is finite and composed of n elements
2. $G : B(E) \to B(E)$ is a contraction mapping whose fixed point is denoted by R^*
3. $R \in B(E)$.

Algorithmic models for computing a fixed point

All elements of R are refreshed: Suppose have the algorithm that updates at stage k ($k \geq 0$) R as follows:

$$R \leftarrow GR. \tag{27}$$

The value of R computed by this algorithm converges to the fixed point R^* of G. This is an immediate consequence of equation (26).

One element of R is refreshed: Suppose we have the algorithm that selects at each stage k ($k \geq 0$) an element $e \in E$ and updates $R(e)$ as follows:

$$R(e) \leftarrow (GR)(e) \tag{28}$$

leaving the other components of R unchanged. If each element e of E is selected an infinite number of times then the value of R computed by this algorithm converges to the fixed point R^*.

One element of R is refreshed and noise introduction: Let $\eta \in \mathbb{R}$ be a noise factor and $\alpha \in \mathbb{R}$. Suppose we have the algorithm that selects at stage k ($k \geq 0$) an element $e \in E$ and updates $R(e)$ according to:

$$R(e) \leftarrow (1-\alpha)R(e) + \alpha((GR)(e) + \eta) \tag{29}$$

leaving the other components of R unchanged.

We denote by e_k the element of E selected at stage k, by η_k the noise value at stage k and by R_k the value of R at stage k and by α_k the value of α at stage k. In order to ease further notations we set $\alpha_k(e) = \alpha_k$ if $e = e_k$ and $\alpha_k(e) = 0$ otherwise.

With this notation equation (29) can be rewritten equivalently as follows:

$$R_{k+1}(e_k) = (1-\alpha_k)R_k(e_k) + \alpha_k((GR_k)(e_k) + \eta_k). \tag{30}$$

We define the history $\mathcal{F}_k$ of the algorithm at stage k as being:

$$\mathcal{F}_k = \{R_0, \dots, R_k, e_0, \dots, e_k, \alpha_0, \dots, \alpha_k, \eta_0, \dots, \eta_{k-1}\}. \tag{31}$$

We assume moreover that the following conditions are satisfied:

1. For every k, we have

$$E[\eta_k | \mathcal{F}_k] = 0. \tag{32}$$

2. There exist two constants A and B such that $\forall k$

$$E[\eta_k^2 | \mathcal{F}_k] \leq A + B \|R_k\|_\infty^2. \tag{33}$$

3. The $\alpha_k(e)$ are nonnegative and satisfy

$$\sum_{k=0}^{\infty} \alpha_k(e) = \infty, \quad \sum_{k=0}^{\infty} \alpha_k^2(e) < \infty. \tag{34}$$

Then the algorithm converges with probability 1 to R^*.

The Q-function as a fixed point of a contraction mapping

We define the mapping $H \colon B(\mathcal{S} \times \mathcal{A}) \to B(\mathcal{S} \times \mathcal{A})$ such that

$$(HK)(s,a) = \underset{w \sim P_w(\cdot|s,a)}{E}[r(s,a,w) + \gamma \max_{a' \in \mathcal{A}} K(f(s,a,w),a')] \tag{35}$$

$\forall (s,a) \in \mathcal{S} \times \mathcal{A}$.

• The recurrence equation (8) for computing the Q_N-functions can be rewritten $Q_N = HQ_{N-1}$ $\forall N > 1$, with $Q_0(s,a) \equiv 0$.

• We prove afterwards that H is a contraction mapping. As immediate consequence, we have, by virtue of the properties algorithmic model (27), that the sequence of Q_N-functions converges to the unique solution of the Bellman equation (9) which can be rewritten: $Q = HQ$. Afterwards, we proof, by using the properties of the algorithmic model (30), the convergence of the Q-learning algorithm.

H is a contraction mapping

This H mapping is a contraction mapping. Indeed, we have for any functions $K, \overline{K} \in B(\mathcal{S} \times \mathcal{A})$:[9]

$$\begin{aligned}
\|HK - H\overline{K}\|_\infty &= \gamma \max_{(s,a)\in\mathcal{S}\times\mathcal{A}} \Big| \underset{w\sim P_w(\cdot|s,a)}{E} \big[\max_{a'\in\mathcal{A}} K(f(s,a,w),a') - \\
&\qquad \max_{a'\in\mathcal{A}} \overline{K}(f(s,a,w),a')\big]\Big| \\
&\leq \gamma \max_{(s,a)\in\mathcal{S}\times\mathcal{A}} \Big| \underset{w\sim P_w(\cdot|s,a)}{E} \big[\max_{a'\in\mathcal{A}} |K(f(s,a,w),a') - \\
&\qquad \overline{K}(f(s,a,w),a')|\big]\Big| \\
&\leq \gamma \max_{s\in\mathcal{S}} \max_{a\in\mathcal{A}} |K(s,a) - \overline{K}(s,a)| \\
&= \gamma \|K - \overline{K}\|_\infty
\end{aligned}$$

[9] We make as additional assumption here that the rewards are strictly positive.

Q-learning convergence proof

The Q-learning algorithm updates Q at stage k in the following way[10]

$$
\begin{aligned}
Q_{k+1}(s_k, a_k) &= (1-\alpha_k)Q_k(s_k, a_k) + \alpha_k(r(s_k, a_k, w_k) + \qquad (36)\\
&\quad \gamma \max_{a \in \mathcal{A}} Q_k(f(s_k, a_k, w_k), a)), \qquad (37)
\end{aligned}
$$

Q_k representing the estimate of the Q-function at stage k. w_k is drawn independently according to $P_w(\cdot|s_k, a_k)$.

[10]The element (s_k, a_k, r_k, s_{k+1}) used to refresh the Q-function at iteration k of the Q-learning algorithm is "replaced" here by $(s_k, a_k, r(s_k, a_k, w_k), f(s_k, a_k, w_k))$.

By using the H mapping definition (equation (35)), equation (37) can be rewritten as follows:

$$Q_{k+1}(s_k, a_k) = (1 - \alpha_k)Q_k(s_k, a_k) + \alpha_k((HQ_k)(s_k, a_k) + \eta_k) \tag{38}$$

with

$$\begin{aligned}
\eta_k &= r(s_k, a_k, w_k) + \gamma \max_{a \in \mathcal{A}} Q_k(f(s_k, a_k, w_k), a) - (HQ_k)(s_k, a_k) \\
&= r(s_k, a_k, w_k) + \gamma \max_{a \in \mathcal{A}} Q_k(f(s_k, a_k, w_k), a) - \\
&\quad \underset{w \sim P_w(\cdot|s,a)}{E} [r(s_k, a_k, w) + \gamma \max_{a \in \mathcal{A}} Q_k(f(s_k, a_k, w), a)]
\end{aligned}$$

which has exactly the same form as equation (30) (Q_k corresponding to R_k, H to G, (s_k, a_k) to e_k and $\mathcal{S} \times \mathcal{A}$ to E).

We know that H is a contraction mapping. If the $\alpha_k(s_k, a_k)$ terms satisfy expression (34), we still have to verify that η_k satisfies expressions (32) and (33), where

$$\mathcal{F}_k = \{Q_0, \ldots, Q_k, (s_0, a_0), \ldots, (s_k, a_k), \alpha_0, \ldots, \alpha_k, \eta_0, \ldots, \eta_{k-1}\}, \tag{39}$$

in order to ensure the convergence of the Q-learning algorithm.

We have:

$$\begin{aligned} E[\eta_k|\mathcal{F}_k] &= \underset{w_k \sim P_w(\cdot|s_k,a_k)}{E}[r(s_k, a_k, w_k) + \gamma \max_{a\in\mathcal{A}} Q_k(f(s_k, a_k, w_k), a) - \\ &\quad \underset{w \sim P_w(\cdot|s_k,a_k)}{E}[r(s_k, a_k, w) + \gamma \max_{a\in\mathcal{A}} Q_k(f(s_k, a_k, w), a)]|\mathcal{F}_k] \\ &= 0 \end{aligned}$$

and expression (32) is indeed satisfied.

In order to prove that expression (33) is satisfied, one can first note that :

$$|\eta_k| \leq 2B_r + 2\gamma \max_{(s,a)\in\mathcal{S}\times\mathcal{A}} Q_k(s,a) \tag{40}$$

where B_r is the bound on the rewards. Therefore we have :

$$\eta_k^2 \leq 4B_r^2 + 4\gamma^2 (\max_{(s,a)\in S\times A} Q_k(s,a))^2 + 8B_r\gamma \max_{(s,a)\in\mathcal{S}\times\mathcal{A}} Q_k(s,a) \tag{41}$$

By noting that

$$8B_r\gamma \max_{(s,a)\in\mathcal{S}\times\mathcal{A}} Q_k(s,a) < 8B_r\gamma + 8B_r\gamma(\max_{(s,a)\in\mathcal{S}\times\mathcal{A}} Q_k(s,a))^2 \tag{42}$$

and by choosing $A = 8B_r\gamma + 4B_r^2$ and $B = 8B_r\gamma + 4\gamma^2$ we can write

$$\eta_k^2 \leq A + B\|Q_k\|_\infty^2 \tag{43}$$

and expression (33) is satisfied. **QED**

References

Pascal Leroy, Pablo G. Morato, Jonathan Pisane, Athanasios Kolios, and Damien Ernst. Imp-marl: a suite of environments for large-scale infrastructure management planning via marl, 2023.

Elia Kaufmann, Leonard Bauersfeld, Antonio Loquercio, Matthias Mueller, Vladlen Koltun, and Davide Scaramuzza. Champion-level drone racing using deep reinforcement learning. *Nature*, 620:982–987, 08 2023.

Lucian Busoniu, Robert Babuska, Bart De Schutter, and Damien Ernst. *Reinforcement learning and dynamic programming using function approximators*. CRC press, 2017.

Richard S Sutton and Andrew G Barto. *Reinforcement learning: An introduction*. MIT press, 2018.

Dimitri Bertsekas. *Dynamic programming and optimal control: Volume I*, volume 4. Athena scientific, 2012.

Csaba Szepesvári. *Algorithms for reinforcement learning*. Springer Nature, 2022.